Insurability of low-probability risks

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Abstract

Why are catastrophic events so difficult to insure? Their extreme severity advocates for a wide-spread use of insurance, yet they are typically excluded from US homeowner policies. It is also puzzling to notice that some low-probability risks such as damages from lightnings are covered under all US homeowner policies whereas others, such as earthquakes and floods are not. Our model explains this puzzle by showing that the effect of aggregate uncertainty, a well-known threat to insurability, is amplified when the individual loss probability is small. Low-probability risks with aggregate uncertainty, such as earthquakes and floods, therefore display lower insurance take-up rates than low-probability risks without aggregate uncertainty such as lightning strikes.

Keywords: low-probability risks, catastrophic risks, insurance, capital cost.

JEL classification: D86, G22, G28, Q54.

1 Introduction

Why are catastrophic events so difficult to insure? Natural and man-made catastrophes such as earthquakes, floods, nuclear accidents and terrorists attacks have severe consequences at the individual level, affect large numbers of people at the same time, but have small individual probabilities of occurrence. The extreme severity of these risks advocates for a wide-spread use of prevention and insurance. Yet, they are typically excluded from US homeowner policies.[1] Private insurance markets are often non-existant and even when specific contracts, benefiting from public subsidies are available, relatively few people purchase them.[2]

It is also puzzling to notice that some low-probability risks such as damages from lightnings are efficiently handled by the insurance sector and covered under the US homeowner policies whereas others, such as earthquakes and floods are not. Our model explains this puzzle by showing that the effect of aggregate uncertainty, a well-known threat to insurability, is amplified when the individual loss probability is low. Low-probability risks with aggregate uncertainty, such as earthquakes and floods, therefore display lower insurance take-up rates than low-probability risk without aggregate uncertainty, such as lightning strikes.

We begin by examining the case without aggregate uncertainty. In this framework, we show that low-probability risks are actually easier to insure than high-probability risks. It is known since Mossin (1968) that expected utility maximizing agents optimally purchase partial coverage when insurance is sold above actuarially fair price. In this case, a decrease in the loss probability has two effects on the demand for coverage. On the one hand, the cost of providing insurance diminishes, which translates into lower premiums for policyholders. On the other hand, the likelihood of receiving the indemnity declines as well. Our contribution in this framework is to show that the cost-reduction effect dominates the likelihood-reduction effect under risk aversion. As the loss probability declines, the ratio of willingness to pay to cost of coverage always increases. In a population heterogeneous in wealth or preferences, take-up rates are therefore predicted to increase as the loss probability decreases.[3] In addition, we show that given the opportunity to do so, people optimally choose higher levels of coverage for low-probability

[1] http://www.iii.org/article/which-disasters-are-covered-by-homeowners-insurance

[2] Kunreuther (1973), Kousky & Kunreuther (2014) and Grislain-Letrémy (2015) document low take-up rates for catastrophe insurance while Cole et al. (2014) and Mobarak & Rosenzweig (2013) try to explain low rates in the micro-insurance industry.

[3] With the exception of the pathological case of Giffen behaviors which, to our knowledge have never been observed in an insurance market.

risks if their index of absolute risk aversion does not decrease too fast with wealth.

These results are in line with Laury et al. (2009), who experiment in the laboratory the effect of a change in the loss probability. A previous experiment by McClelland et al. (1993) also found a decreasing mean ratio of willingness to pay to expected indemnity, indicating that people tend to be willing to pay higher loadings for low-probability risks than for high-probability risks.

These observations, however, are at odds with the low take-up rates for disaster risks. Our explanation is that the risks for which underinsurance is most prevalent display substantial aggregate uncertainty. Natural and man-made catastrophes feature geographically correlated individual losses which translates into aggregate loss uncertainty, even within very large pools of policyholders. In order to remain solvent, the insurance provider must either raise prohibitively high levels of premiums or more realistically, it must have access to capital to fill the gap between the premiums raised and the amount of claims due in case of catastrophe. The cost of allocating this capital to a specific line of business[4] depends on the size of the potential worst case *aggregate* loss. This fixed cost causes the loading to increase when the loss probability decreases and eventually discourages people from purchasing coverage. In a world where agents are heterogeneous in wealth or preferences, a decrease in probability may therefore induce lower take-up rates. This prediction is much more in line with the observations of the take-up rates in various settings.

Several literatures have attempted to explain the low take-up rates for disaster coverage. Kunreuther & Slovic (1978), Hertwig et al. (2004) and Kunreuther et al. (2001) rely on departures from the expected utility paradigm, arguing that low-probabilities are more difficult to process than high probabilities. In the expected utility framework, Kunreuther et al. (2001) show that search costs may generate a probability threshold below which coverage is not purchased while Coate (1995) shows how government relief can crowd-out the demand for insurance. Raschky et al. (2013), Kousky et al. (2013) and Grislain-Letrémy (2015) find empirical support for this hypothesis in various countries. Finally, in the context of micro-insurance Cole et al. (2014) suggests that learning is an important determinant of the demand for insurance.

Overall, much less attention has been devoted to the supply side of the market. Kunreuther et al. (1995) explain that ambiguity may provide a rationale for high disaster insurance premiums. In the context of micro-

[4]See Zanjani (2002) and Froot (2001).

insurance markets, Mobarak & Rosenzweig (2013) argue that basis risk can explain low take-up rates.

Our contribution to the literature is twofold. First, we provide a simple, yet general framework, that provides an alternative explanation to the observed low take-up rates for disaster risks.[5] Second, we explain this apparently puzzling observation that some low-probability events are well insured while others are not. It is in fact the combination of aggregate loss uncertainty with low-probability that makes earthquakes, floods and terrorism risks difficult to insure.

2 A model with aggregate loss uncertainty

This section lays out the framework for the analysis of the insurability of low-probability events. Our model permits the analysis of correlated risks and encompasses the standard independent losses model as a particular case.

We consider a population of unit mass, in which each individual may or may not be affected by a loss of size L. Similarly to Charpentier & Le Maux (2014), the share of the population affected by a loss is a random variable $\tilde{q}$. It is distributed with cumulative distribution function

$$F(q) = \int_0^q dF(q) \qquad \forall q \in (0,1),$$

with expected value p and variance σ^2. From the insurer's standpoint, there is therefore a continuum of states of the world, that we call aggregate states, characterized by a fraction q of the population being affected by a loss. Notice that almost all lines in the insurance business feature some kind of aggregate loss uncertainty. Finding the distribution of the expected number of claims, and therefore of the fraction of the policyholders who will file a claim, constitutes an important task of actuaries.

Conditionally on the occurrence of a particular aggregate state of the world, losses are independently distributed across people. Calling $\tilde{x}_i$ the random loss of agent i, the probability that an individual faces a loss is therefore

$$\begin{aligned} p &= \int_0^1 \mathbb{P}(\tilde{x}_i = L | \tilde{q} = q) dF(q) \\ &= \int_0^1 q dF(q). \end{aligned}$$

[5] Jaffee & Russell (1997) discuss the problem of capital allocation cost but provide no formal model while Kousky & Cooke (2012) have a simulated model, calibrated to analyze flood insurance coverage in Broward County, Florida.

The individual loss is therefore a Bernouilli random variable with parameter p equal to the expected value of $\tilde{q}$. Losses are in general not independent random variables. For two individuals i and j

$$\mathbb{C}ov(\tilde{x}_i, \tilde{x}_j) = \mathbb{E}(\tilde{x}_i \tilde{x}_j) - \mathbb{E}(\tilde{x}_i)\mathbb{E}(\tilde{x}_j).$$

Since we have an infinite number of agents[6]

$$\tilde{x}_i \tilde{x}_j = \begin{cases} L^2 & \text{with probability} \quad \int_0^1 q^2 dF(q) \\ 0 & \text{with probability} \quad 1 - \int_0^1 q^2 dF(q), \end{cases}$$

the covariance between two losses can be written as

$$L^2 \int_0^1 q^2 dF(q) - (L \int_0^1 q dF(q))^2 > 0. \tag{1}$$

We show in Appendix (6.1) that the coefficient of correlation between two losses can be written

$$\delta = \frac{\sigma^2}{p(1-p)},$$

and that $\sigma^2 < p(1-p)$ which guarantees that $\delta < 1$. Given a fixed expected value p, increasing the variance of the distribution F always increases the correlation coefficient. For a given σ^2, a decrease in p results in an increase in δ if $p < 1/2$.

This model of loss exposure nests the usual independent losses model as a specific case in which the fraction q is known. The probability density function becomes

$$dF(q) = \begin{cases} 1 & \text{for some} \quad q \in (0,1) \\ 0 & \text{otherwise.} \end{cases}$$

Individual losses are usual Bernoulli random variables with parameter $p = q$ and since $\sigma^2 = 0$, the coefficient of correlation δ is null.

3 Insuring low-probability risks in the absence of aggregate uncertainty

In this section, we show that in the baseline insurance framework with independent losses, low-probability events are in fact more likely to be insured

[6]An infinite number of agents enables us to find a simple expression for the coefficient of correlation since it implies that all the aggregate uncertainty comes from the dependence between the individual losses. But none of our results relies on this assumption.

than higher probability events. The intuition behind this result is that individual's willingness to pay is concave in the probability, while the cheapest feasible contract is linear. Therefore, if a risk is insurable somewhere over the parameter space, there is a probability threshold below which it is insurable and above which it is not.

We consider risk-averse agents with a twice continuously differentiable and concave utility function $u(x)$ and initial wealth w. $A(x) = -\frac{u''(x)}{u'(x)}$ denotes the Arrow-Pratt index of absolute risk aversion.

The insurance provider can be private or public. It is represented by a risk-neutral agents that sells an amount of coverage $\tau \in [0, L]$ at a premium α. Following Raviv (1979) we assume that, in addition to the payment of the indemnity τ, the insurance provider faces a cost $c(\tau)$ with

$$c(0) = 0, \quad c'(0) = b > 0, \quad c'(\tau) > 0 \quad \text{and} \quad c''(\tau) \geq 0,$$

which represents the various expenses associated with the payment of an indemnity τ to all affected agents. It can be interpreted as an administrative cost, as a cost of expertise, or more broadly as a dead-weight loss, resulting either from an asymmetry of information between the insurance company and the policy holder, or by imperfect competition. The actuarial and insurance literatures often make the simplifying assumption that the marginal cost $c'(\tau)$ is equal to a constant λ called the loading factor. In this case, the dead-weight cost is simply a fraction of the indemnity. We call loading the ratio $\alpha/p\tau$ premium to expected indemnity. If $c'(\tau) = \lambda$, the loading is $1 + \lambda$ and we have $c''(\tau) = 0$, which is indeed a particular case of our model.

A necessary condition for the insurance to provide such a contract is that its expected profit is positive. This motivates the following definition

Definition 1 *A contract is called feasible if and only if the insurance provider can realize at least a zero expected profit.*

In the absence of aggregate loss uncertainty, a contract is therefore feasible if and only if

$$\alpha \geq p\tau + pc(\tau).$$

The first notion of insurability that we develop is the following

Definition 2 *A risk is strongly insurable at a level τ if and only if individuals are willing to purchase the proposed level of coverage τ of some feasible contract.*

In order to know whether a risk is strongly insurable or not, it is sufficient to verify that individuals are willing to purchase the zero expected profit contract, for which the insurer breaks even. If they reject the zero expected profit feasible contract, agents will also reject all the other more expansive contracts, and if they accept the zero expected profit feasible contract, then the risk is insurable. In the remaining of the paper, we use the word contract to mean zero expected profit contract.

Independently of the supply side constraints, the highest price $C(p,\tau)$ that an individual would pay for a level of coverage τ is given by

$$pu(w-L+\tau-C)+(1-p)u(w-C)=pu(w-L)+(1-p)u(w). \tag{2}$$

First notice that $C(0,\tau)=0$ and $C(1,\tau)=\tau$. The willingness to pay for the coverage of a a zero probability event is zero and the willingness to pay for the coverage of a sure event is just the coverage itself. Total differentiation of (2) gives

$$C_p'(p,\tau)=\frac{u(w)-u(w-L)-[u(w-C)-u(w-L+\tau-C)]}{pu'(w-L+\tau-C)+(1-p)u'(w-C)}\geq 0 \quad \forall\tau\leq L,$$

and

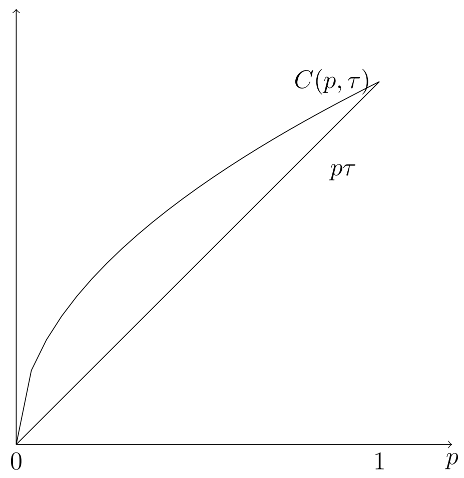
$$\begin{aligned} C_{pp}''(p,\tau) = & - 2C_p'\frac{u'(w-L+\tau-C)-u'(w-C)}{pu'(w-L+\tau-C)+(1-p)u'(w-C)} \\ & + (C_p')^2\frac{pu''(w-L+\tau-C)+(1-p)u''(w-C)}{pu'(w-L+\tau-C)+(1-p)u'(w-C)}\leq 0 \quad \forall\tau\leq L, \end{aligned}$$

where C_p' and C_p'' represent the first and second order partial derivative with respect to p. The agent's willingness to pay C is therefore increasing and concave in the probability of loss p. Figure 1a represents the agent's willingness to pay and the cost of coverage as a function of the loss probability p for the case $c(\tau)=0$. When coverage is sold at an actuarially fair price, the agent is always willing to purchase it. The surplus she derives from the transaction however, may vary with the loss probability.

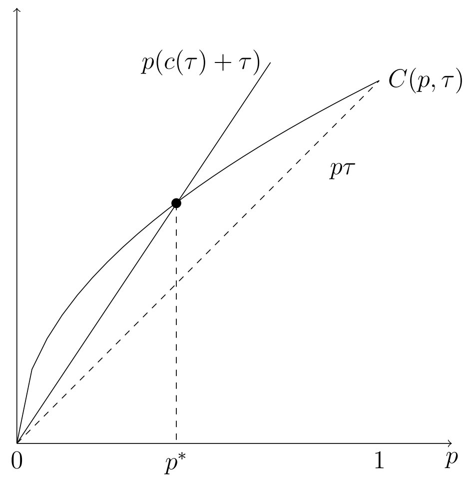
Figure 1b represents the case where coverage is available at a cost higher than the actuarially fair price. For values of p lower than a threshold p^*, willingness to pay is above the price of coverage. An insurance market should therefore emerge in this case. For values of p higher than p^*, willingness to pay is below the price of the zero expected profit contract, resulting in the absence of any market.

Proposition 1 *In the absence of aggregate loss uncertainty, a risk is strongly insurable at level τ if and only if the individual probability of loss p is below a threshold p^*, where p^* is such that*

$$C(p^*,\tau)=p^*\tau+p^*c(\tau).$$



(a) Actuarially fair coverage



(b) Non-actuarially fair coverage

For a given level of coverage, low-probability events are therefore more likely to be covered than high-probability events.

Laury et al. (2009) investigate in an experiment how insurance purchase decisions evolve with p. They observe that the fraction of their sample that purchases full-coverage decreases with the probability of loss p, given a constant expected loss and loading. Proposition (1) confirms that this should indeed be the case but delivers an even stronger prediction: the fraction of a population that purchases coverage should increase as p decreases even when the loss L is fixed.

To see this, assume that the population is heterogeneous in wealth and preferences. An agent i endowed with wealth w^i and utility function u^i purchases a feasible contract providing coverage τ if and only if

$$C_i(p,\tau) \geq p\tau + pc(\tau) \tag{3}$$

Each individual therefore has a probability threshold p_i^* above which she stops purchasing insurance. The distribution of wealth and preferences generates a distribution G over the thresholds p_i^*. The fraction of the population that purchases insurance is $\mathbb{P}(p_i^* \geq p) = 1 - G(p_i^*)$, which is decreasing in p.

An insurance contract providing a level of coverage τ is therefore more likely to be purchased when the probability of loss p is smaller. In this sense, low-probability events are more insurable than higher-probability events. However, the exogeneity of the level of coverage τ may appear as a limit to this analysis. If people have control over the level of coverage, the decision is not a take-it-or-leave-it problem anymore.

With this idea in mind, we propose the following notion of insurability:

Definition 3 *A risk is weakly insurable if and only if individuals are willing to purchase a positive amount of coverage of some feasible contract.*

This notion of insurability is weaker in the sense that a strongly insurable risk is necessarily weakly insurable. If a person agrees to purchase a level of coverage τ rather than no insurance when the loss probability is p, she would never optimally choose $\tau = 0$. However, she may select a level of coverage lower than τ to reduce the cost of insurance.

With endogenous coverage, the agent solves

$$\begin{aligned} \max pu(w - L + \tau - \alpha) + (1-p)u(w-\alpha) \\ \text{s.t.} \quad \alpha = p\tau + pc(\tau) \\ \tau \geq 0. \end{aligned} \tag{4}$$

The first order condition of this problem is

$$[1 - (1 + c'(\tau))p)]u'(w_1) = (1-p)(1+c'(\tau))u'(w_2), \tag{5}$$

in which $w_1 = w - L + \tau - p\tau - pc(\tau)$ and $w_2 = w - p\tau - pc(\tau)$ are the levels of wealth in the loss and no-loss states. It is easy to check that $c'(\tau) > 0$ implies that the agent chooses partial coverage, so that $\tau < L$ at any interior solution.

Individuals purchase a positive amount of coverage if and only if

$$(1-p)(1+b)u'(w) < [1-(1+b)p]u'(w-L). \tag{6}$$

Re-arranging the terms of (6) yields the following Proposition.

Proposition 2 *In the absence of aggregate loss uncertainty, a risk is weakly insurable if and only if its probability p is such that*

$$p < \frac{1}{1+b} - \frac{b}{1+b}\frac{u'(w)}{u'(w-L)-u'(w)}. \tag{7}$$

For a given p, an event is uninsurable when the marginal cost of coverage $b = c'(0)$ is too high or the size of the loss L is not sufficiently large to generate a significant difference between marginal utility in the loss state $u'(w-L)$ and marginal utility in the no-loss state $u'(w)$. Proposition (2) stresses once more that, in the absence of aggregate loss uncertainty, low-probability events are easier to insure than high-probability events.

In addition, Proposition 3 and its corollary give conditions under which a strictly positive optimal coverage increases when the probability p decreases.

When the loss probability diminishes, two effects interact. On the one hand, the risk of experiencing the loss L diminishes, lowering the incentive to pay the dead-weight cost $c(\tau)$. When p diminishes, a higher level of risk aversion is therefore required to justify the purchase of coverage. On the other hand, the dead-weight marginal cost $pc'(\tau)$ diminishes, making insurance more attractive at the margin. The total effect on optimal coverage cannot be signed for any utility function, but the following proposition and corollary enables to identify some interesting and realistic cases.

Proposition 3 *In the absence of aggregate loss uncertainty, the optimal coverage τ of a weakly insurable risk is strictly decreasing in p if and only if*

$$A(w_1) - A(w_2) < \frac{c'(\tau)}{[\tau + c(\tau)](1-p)[1-(1+c'(\tau))p]}$$

at the optimum.

Proof. The proof is given in Appendix 6.2 ■

The left-hand side of the inequality is positive when $A(w_1) \geq A(w_2)$, while the right-hand side is positive when $1-(1+c'(\tau))p > 0$, which is a necessary condition for an interior solution. Since $w_1 < w_2$, the condition is trivially satisfied for any increasing or constant absolute risk aversion functions (IARA or CARA). For the class of decreasing absolute risk aversion (DARA), the condition puts an upper bound on the variation of risk aversion between the loss and the no-loss state.

The empirical literature most often fails to reject DARA as a realistic hypothesis such as in Guiso & Paiella (2008) and Levy (1994). In addition, Rabin (2000)'s calibration theorem, showing that aversion to small-stake gambles implies rejection of gambles that people take in their lives, can be interpreted as a rejection of the CARA (and even more of the IARA) hypothesis. We would therefore like to know whether classical utility functions, satisfying the DARA property, also feature an optimal coverage decreasing in the probability p. This is the purpose of Corollary 3.1 that deals with the case of Harmonic Absolute Risk Aversion (HARA) functions. We define a HARA utility function as in Gollier (2004)

$$u(x) = \zeta\Big(\eta + \frac{x}{\gamma}\Big)^{1-\gamma},$$

whose domain is such that $\eta + \frac{x}{\gamma} > 0$ and the condition $\zeta\frac{1-\gamma}{\gamma} > 0$ guarantees that the function is indeed increasing and concave. The coefficient of absolute risk aversion is

$$A(x) = \Big(\eta + \frac{x}{\gamma}\Big)^{-1}. \tag{8}$$

Except for the limit case $\gamma \to +\infty$, the HARA functions satisfy the DARA property when $\gamma > 0$, which makes them appealing with respect to the literature discussed previously.

Corollary 3.1 *If a risk-averse agent has preferences represented by a HARA utility function with $\gamma \geq 1$, then the optimal coverage τ of a weakly insurable risk is strictly decreasing in p in the absence of aggregate loss uncertainty.*

Proof. The proof is given in Appendix 6.3 ■

The HARA class nests two of the most widely used classes of utility functions. The Constant Absolute Risk Aversion (CARA) is obtained when $\gamma \to +\infty$. Solving the differential equation (8) yields the specification of this function

$$u(x) = -\eta \exp\left(-\frac{1}{\eta}x\right).$$

The second class of functions within the HARA class is the set of Constant Relative Risk Aversion (CRRA) which is obtained for $\eta = 0$. Solving (8) in this case yields

$$u(x) = \frac{x^{1-\gamma}}{1-\gamma}. \tag{9}$$

If people's preferences can be represented by a CARA or by a CRRA utility function with $\gamma \geq 1$, then their optimal coverage is always a decreasing function of p. For the CRRA case, Szpiro (1986) and Barsky et al. (1997) find values of relative risk aversion respectively between 1.2 and 1.8 for the first and 4.17 for the second. According to Gollier (2004) (p.69), the "range of acceptable values of relative risk aversion [is] $[1, 4]$". A complete survey of the literature on risk preferences elicitation would reveal some estimates below one, such as Chetty (2006), but overall our (sufficient but not necessary) condition $\gamma \geq 1$ seems a very plausible assumption.

Finally, Louaas & Picard (2014) have shown the following proposition

Proposition 4 *In the absence of aggregate loss uncertainty and if $u'(w - L) > (1+b)u'(w)$, the optimal coverage converges toward a positive limit τ defined as*

$$u'(w - L + \tau) = (1 + c'(\tau))u'(w).$$

It may sound surprising that the agent is willing to pay a positive loading factor for a risk whose probability tends to zero, but the pricing rule $\alpha = p\tau + pc(\tau)$ implies that the premium tends to zero with the probability. The agent therefore receives an indemnity with an infinitesimal probability whose price also becomes infinitesimal.

Figure 2 gives an illustration of the previous results in the (w_1, w_2) space. Point A represents the optimal lottery at the highest probability p for which the insured chooses a positive amount of coverage. The thick curve represents the locus of optimal lotteries as p diminishes and Point B is the limit optimal

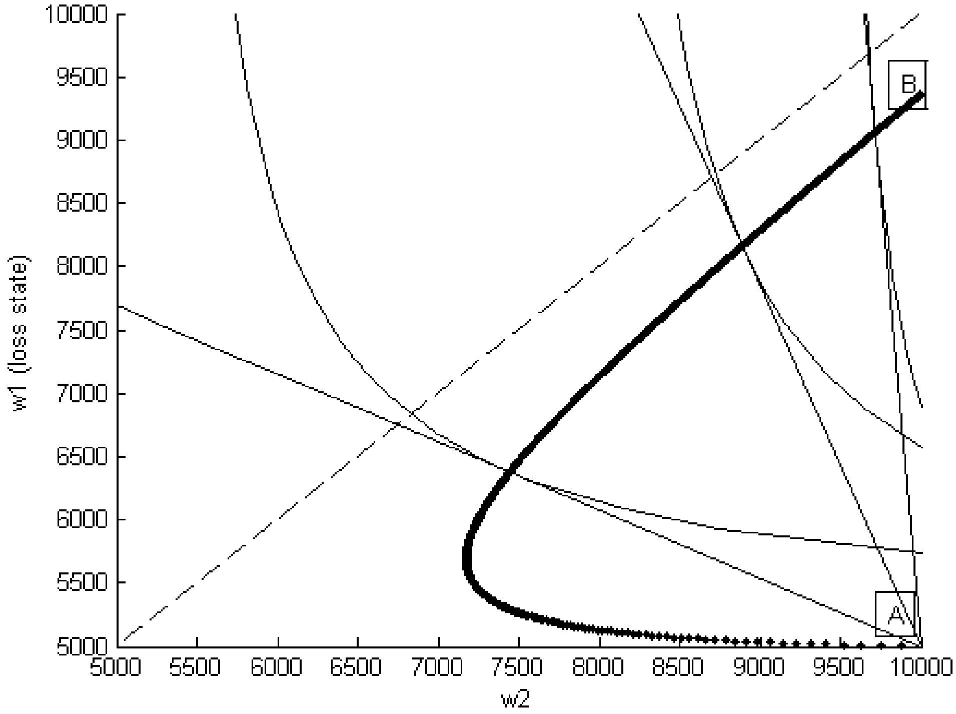


Figure 2: $w = 10000, L = 5000, u(x) = \frac{x^{-3}}{3}$

lottery when $p \to 0$. In agreement with Proposition 3, the optimal lottery gets closer from the 45 degree line as p decreases, until it (almost) reaches point B when $p \to 0$.

Our theoretical results indicate that under reasonable assumption on individual's preferences, low-probability events should be more insured than higher probability events. Gollier (1997) shows that the deductible chosen by a risk averse agent is decreasing in p when p is sufficiently small and the agent's utility function features IARA, which is equivalent[7] to purchase more insurance when the probability is very small. Our results are more general since they apply everywhere on the domain of p where the agent purchases a positive amount of coverage. In addition, we are able to accommodate some of the more realistic DARA utility functions.

Low-probability and high-stake risks are characterized by a large potential loss L. In order to compare insurance choice for these risks to choices concerning more traditional risks, it may be useful to fix the expected loss, as

[7]In a model where agents only face two states: loss or no loss, deductible and partial coverage are strictly equivalent.

in Laury et al. (2009), so that the loss increases as its probability diminishes. In this case and independently of any assumption on individual's preferences, agents purchase more coverage as the probability of loss decreases.

Proposition 5 *In the absence of aggregate loss uncertainty, and if the expected loss pL remains constant, the optimal coverage is a decreasing function of the probability p for any risk averse individual.*

Proof. The proof is given in Appendix 6.4 ■

The requirement that expected loss be fixed makes it easier for the condition of proposition (3) to hold. Indeed as p diminishes, the loss becomes larger, providing agents with additional incentives to purchase insurance. This in turn reduces the gap between wealth in the non loss state w_1 and wealth in the loss state w_2.

This section has shown that in the absence of aggregate loss uncertainty, low-probability risks are easier to insure than high-probability risks. This may explain why typical homeowners insurance cover perils as unlikely as lightening strikes. In contrast, low-probability correlated risks such as earthquakes, flooding, windstorms, nuclear hazards and acts of wars are typically excluded from homeowner policies. The next section provides an explanation as to why low-probability correlated risks are also difficult to insure.

4 Insuring low-probability risks in the presence of aggregate uncertainty

This section considers the case of correlated risks. We relax the assumption that the fraction q of the population affected by the loss is known. Instead, $\tilde{q}$ is a random variable with cumulative distribution function $F(q)$, expected value p and variance σ^2. Because the insurance company raises the premiums ex-ante, the total amount of premium raised is unlikely to match the amount required to pay-off all claims. When it raises a premium α, it must therefore have access to an amount of capital per policy k such that

$$k = \bar{q}[\tau + c(\tau)] - \alpha, \tag{10}$$

where $\bar{q}$ is the highest fraction of affected policyholders with a non-zero measure. This capital can be financed with an internal reserve of liquidities, with some form of reinsurance contract[8] or by borrowing.[9] In any case, the availability of this capital comes at a cost per monetary unit called r. When k is a reserve, r represents the foregone investment opportunities associated with having to keep the reserve. If the insurance company is private, r can be measured as the internal rate of return of the insurance company. If the insurance provider is public, r can be seen as the cost of public funds. When k is a reinsurance or an alternative risk transfer contract, r is simply the rate of return paid to the seller of the contract. The zero expected profit feasible contract is therefore such that

$$\alpha - (\tau + c(\tau)) \int_0^1 q dF(q) - kr = 0.$$

Replacing k by its expression (10) yields the price of the contract

$$\alpha = \frac{\tau + c(\tau)}{1 + r}(p + r\bar{q}).$$

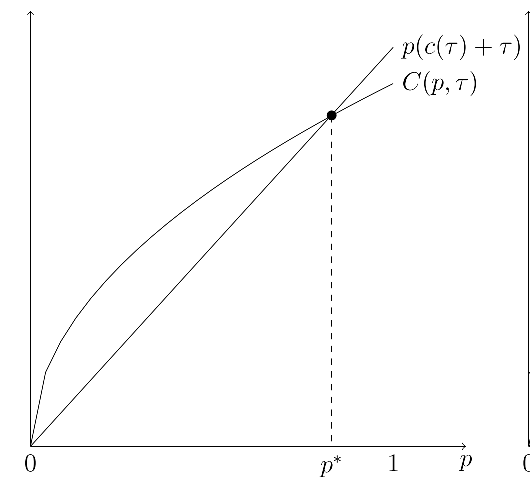
In the absence of aggregate loss uncertainty, $\bar{q} = p$ and we find a lower premium $\alpha = p\tau + pc(\tau)$ which corresponds to the premium derived in the previous section. Aggregate loss uncertainty therefore generates an additional cost determined by the cost of capital allocation and by the tail of the distribution.[10]

Figure 3b displays the highest price that an agent is willing to pay $C(p, \tau)$ and the price of the zero expected profit feasible contract in the presence of aggregate loss uncertainty. The two curves intersect at $\underline{p}$ and $\overline{p}$, hence defining three zones on the $[0, 1]$ interval. Coverage is purchased on the domain $[\underline{p}, \overline{p}]$. Below $\underline{p}$, willingness to pay falls below the cost of insurance, which explains the failure to insure low-probability correlated risks.

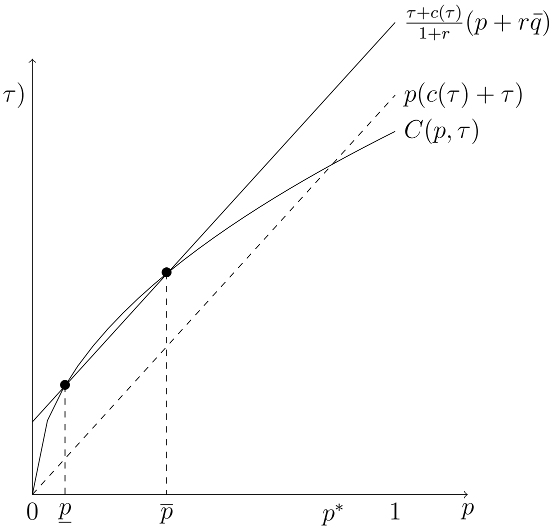
[8]Insurance providers traditionally rely on reinsurance companies to transfer the layers of risk they do not wish to retain. The past two decades have also seen the emergence of Insurance Linked Security (ILS) markets, enabling insurance and reinsurance providers to transfer layers of risk to the financial markets.

[9]This is particularly true for public insurance providers, who have greater access to capital through fiscal policy.

[10]In practice, insurance companies are subject to regulatory solvency constraints that impose a probability of default lower than an exogenously defined level η. An exogenously defined probability of default would require the insurance provider to have access to a lower level of capital $k = F^{-1}(1 - \eta)[\tau + c(\tau)] - \alpha$, which would translate in a lower premium $\alpha = \frac{\tau + c(\tau)}{1+r}(\int_0^{F^{-1}(1-\eta)} qdF(q) + rF^{-1}(1 - \eta))$. Taking this possibility of default into account would not modify our analysis, as long as the probability of default is not taken into account by the policyholders. For an in-depth analysis of endogenous default probability, see Charpentier & Le Maux (2014).



(a) Non-actuarially fair coverage with no aggregate loss uncertainty



(b) Non-actuarially fair coverage with aggregate loss uncertainty

Proposition 6 *If a risk is strongly insurable at some probability level, then in the presence of aggregate loss uncertainty, there exist a probability threshold $\underline{p}$, below which the risk becomes uninsurable (in the strong sense). $\underline{p}$ is given by*

$$\underline{p} = \min\{p | C(p,\tau) = \frac{\tau + c(\tau)}{1+r}(p + r\bar{q})\}$$

This result underlies the fact that in the presence of aggregate loss uncertainty, small-probability risks are indeed more difficult to insure because they are more costly.

As in the previous section, we now proceed to endogenize the level of coverage τ. People are now able to lower the coverage as the probability becomes too small, hence mitigating the increase in the loading $\alpha/p\tau$. It is therefore a priori not clear whether uninsurability will arise for low-probability events in this context. The agent's program is

$$\begin{aligned} \max pu(w + \tau - \alpha - L) + (1-p)u(w-\alpha) \\ \text{s.t.} \quad \alpha = \frac{\tau + c(\tau)}{1+r}(p + r\bar{q}) \\ \tau \geq 0, \end{aligned}$$

and the associated first order condition of is

$$p[1 - \frac{1 + c'(\tau)}{1+r}(p + r\bar{q})]u'(w_1) = (1-p)\frac{1 + c'(\tau)}{1+r}(p + r\bar{q})u'(w_2).$$

The problem has an interior solution if and only if[11]

$$Q(p) = -p^2[u'(w-L)-u'(w)]+p[\frac{1+b}{1+r}u'(w-L)+r\bar{q}(u'(w)-u'(w-L))-u'(w)]-r\bar{q}u'(w) > 0$$

Remark that the left-hand side of this last inequality is quadratic in p with a negative intercept. Figure 4 represents an example of this quadratic form. There may or may not exist an interval of p where the inequality is satisfied (where the agent purchases a positive amount of coverage), depending on the relative values of the parameters, but when such an interval exists, it is delimited downward by a lower bound p^{**}.

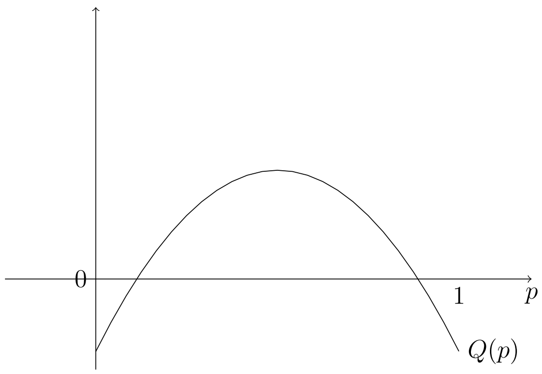


Figure 4: $Q(p)$

Proposition 7 *If a risk is weakly insurable at some probability level, then in the presence of aggregate loss uncertainty, there exist a probability threshold p^{**}, below which the risk becomes uninsurable (in the weak sense). p^{**} is given by*

$$p^{**} = \min\{p|Q(p) = 0\}$$

This results emphasizes that even when the agent is free to choose the level of coverage τ, there is a probability threshold p^{**} below which she will stop purchasing insurance. This also implies the following Corollary

Corollary 7.1 *In the presence of aggregate loss uncertainty, the optimal coverage tends to zero with the probability p.*

This contrasts with Proposition (4) of the previous section. In the presence of aggregate loss uncertainty, the premium α has a component that does not

[11] From the first order condition, we obtain $p[1-\frac{1+b}{1+r}(p+r\bar{q})]u'(w-L)-(1-p)\frac{1+b}{1+r}(p+r\bar{q})u'(w) > 0$.

vary with p. As p becomes very small the cost-reduction effect of a decrease in p vanishes but the likelihood-reduction effect remains and eventually dominates completely, leading the agent to stop purchasing insurance.

Figure 5 shows a calibrated example in the simple case where $\tilde{q}$ takes value $q = 0.5$ with probability $\pi = dF(0.5)$ and 0 with probability $1 - \pi$. The individual probability of loss is $p = \pi q$. The locus of optimal lotteries as the probability π varies is represented by the set of dark dots. When $\pi = 1$, $p = 0.5$ is too large for the agent to purchase any coverage. The optimal lottery is therefore $(w, w - L)$, which is located at the extreme south-east part of the graph. As π diminishes, the cost-reduction effect dominates, driving the optimal lottery closer from the 45 degree line. At a certain point however, the cost-reduction effect becomes weaker and the likelihood-reduction effect dominates, leading coverage down to zero in the limit. The optimal lottery returns to its original $(w, w - L)$ position.

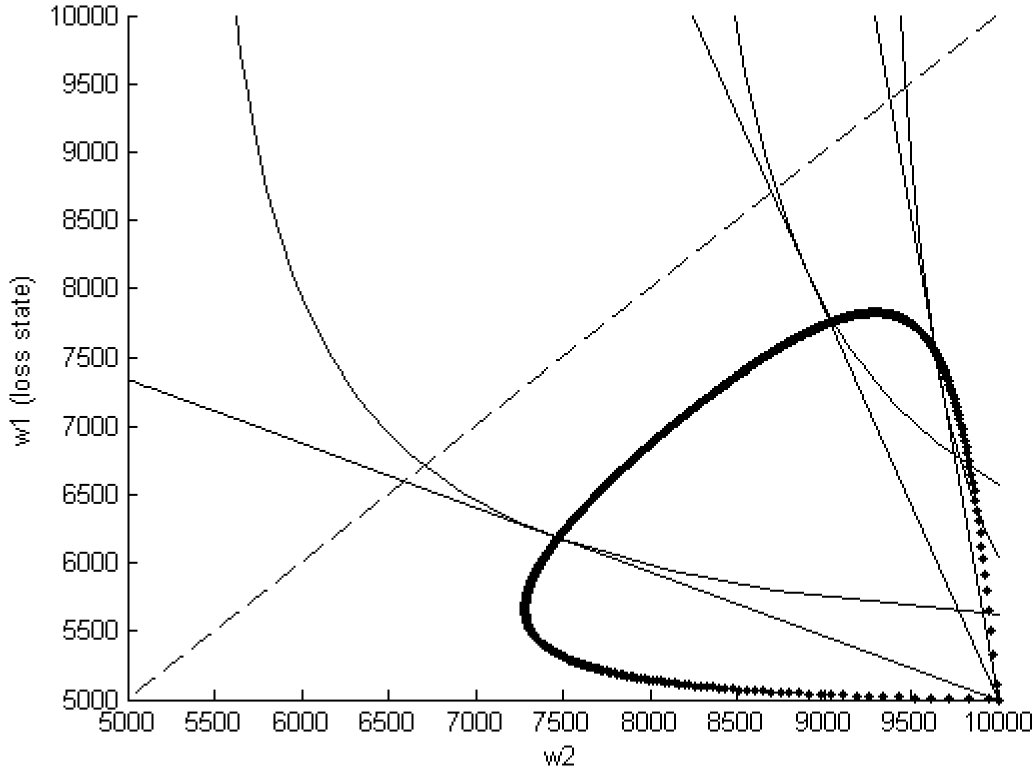


Figure 5: $w = 10000, L = 5000, u(x) = \frac{x^{-3}}{3}, r = 0.05, \lambda = 0.3, q = 0.5$

The actuarial and insurance literatures often use the loading $\alpha/p\tau$ as a measure of the cost of insurance. Apart from the exotic case of Giffen behaviors,[12] which, to our knowledge, has never been observed in insurance markets, the higher the loading, the lower the demand for insurance coverage.

Proposition 8 *In the presence of aggregate loss uncertainty, the loading* $\frac{\tau+c(\tau)}{\tau(1+r)}(1+\frac{r\bar{q}}{p})$ *is a decreasing function of* p *and diverges to* $+\infty$ *as* $p \to 0$.

In the absence of aggregate loss uncertainty, or if the cost of capital allocation is null, the loading $1 + c(\tau)/\tau$ does not depend on the probability p. In this case, the simplifying assumption $c'(\tau) = \lambda$ gives an accurate and useful approximation of the price of insurance. However, it becomes less accurate when, holding other factors constant, the loss probability p becomes small. Empirical studies on the demand for insurance should therefore take this effect into account. Although we have little doubt that the traditional explanations of low take-up rates have a significant importance in the decision to purchase insurance, their effects would be overestimated should one omit to control for variations in the loading factor induced by changes in the loss probability.

5 Conclusion

In this paper, we show that the low insurance take-up rates for low-probability risks can be explained by the presence of aggregate uncertainty. Low probability risks without aggregate uncertainty, are easy to insure because people's willingness to pay, expressed in terms of loading, decreases with the loss-probability. On the contrary, low-probability risks with aggregate uncertainty can be very difficult to insure as the cost of providing coverage, expressed in terms of loading, explodes as the loss probability becomes small. This explains why risks such as lightnings are covered under US homewner policies, while earthquakes, floods and nuclear accidents are typically excluded.

It is also interesting to remark that acts of terrorism were included in standard US homeowner policies until early 2012. One month after the 9/11 attacks, the Insurance Services Office filed a request to exclude acts of terrorism from standard US homeowner policies (Kunreuther & Michel-Kerjan (2005)). In a very similar way, the supply of coverage against losses due to earthquakes dramatically dropped after the 1994 Northridge earthquake that devastated California and generated more than $24 billions in insured losses.

These facts suggest that supply side explanations account for a great share of the insurability problem for low-probability events. The explanation we propose is that the occurrence of large catastrophes changes the way actors

[12] See Briys et al. (1989)

of the insurance industry perceive the risks they insure. The 9/11 attacks and the Northridge earthquake made insurers realize that the worst case scenario was even worst than what they had imagined so far. The impact on the insurability of low-probability events is particularly salient because even small premium increase may generate important changes on the implied loading factor that people have to pay for coverage.

Public-Private Partnership insurance structures, such as the Californian Earthquake Authority, were often created in the aftermath of large catastrophes to face the unavailability of insurance policies at reasonable prices. Their goal is precisely to reduce the cost of capital allocation by aggregating the risk at the highest possible level and by smoothing the shocks across time.

6 Proofs

6.1 Correlation

Remark that

$$q^2 dF(q) = (q-p)^2 dF(q) - p^2 dF(q) + 2pq dF(q).$$

Hence

$$\begin{aligned} L^2 \int_0^1 q^2 dF(q) &= L^2 \int_0^1 (q-p)^2 dF(q) - p^2 dF(q) + 2pq dF(q) \\ &= L^2[\sigma^2 - p^2 + 2p^2] \\ &= L^2[\sigma^2 + p^2]. \end{aligned}$$

Using this expression in (1) yields

$$\begin{aligned} \mathbb{C}ov(\tilde{x}_i, \tilde{x}_j) &= L^2[\sigma^2 + p^2] - p^2 L^2 \\ &= L^2 \sigma^2 \end{aligned}$$

Since $\tilde{x}_i$ and $\tilde{x}_j$ have the same variance equal to $p(1-p)L^2$, the coefficient of correlation between two individual losses writes

$$\delta = \frac{\sigma^2}{p(1-p)}$$

To show that $\delta \leq 1$, remark that

$$\begin{aligned} \sigma^2 &= \int_0^1 (q-p)^2 dF(q) \\ &= \int_0^1 q^2 dF(q) - p^2 \\ &< p - p^2. \end{aligned}$$

6.2 Proposition 3

We consider the case of an interior solution. From the first order condition (5), we obtain

$$\begin{aligned} \frac{d\tau}{dp} &= \frac{(1+c'(\tau))[u'(w_1) - u'(w_2)]}{(1-p)p(1+c'(\tau))^2 u''(w_2) + [1-(1+c'(\tau))p]^2 u''(w_1) - c''(\tau)[pu'(w_1) + (1-p)u'(w_2)]} \\ &+ \frac{[\tau + c(\tau)]\Big([1-(1+c'(\tau))p]u''(w_1) - (1-p)(1+c'(\tau))u''(w_2)\Big)}{(1-p)p(1+c'(\tau))^2 u''(w_2) + [1-(1+c'(\tau))p]^2 u''(w_1) - c''(\tau)[pu'(w_1) + (1-p)u'(w_2)]}. \end{aligned}$$

Since u is concave and c convex we have

$$\begin{array}{c} \frac{d\tau}{dp} < 0 \\ \Leftrightarrow \\ u'(w_1) + \frac{\tau+c(\tau)}{1+c'(\tau)}(1-(1+c'(\tau))p)u''(w_1) > u'(w_2) + \frac{\tau+c(\tau)}{1+c'(\tau)}(1-p)(1+c'(\tau))u''(w_2) \\ \Leftrightarrow \\ u'(w_1)[1 - \frac{\tau+c(\tau)}{1+c'(\tau)}(1-(1+c'(\tau))p)A(w_1)] > u'(w_2)[1 - \frac{\tau+c(\tau)}{1+c'(\tau)}(1-p)(1+c'(\tau))A(w_2)]. \end{array} \tag{11}$$

Using the first order condition (5), this last inequality holds if and only if

$$1-\frac{\tau+c(\tau)}{1+c'(\tau)}(1-(1+c'(\tau))p)A(w_1) > \frac{1-(1+c'(\tau))p}{(1-p)(1+c'(\tau))}[1-\frac{\tau+c(\tau)}{1+c'(\tau)}(1-p)(1+c'(\tau))A(w_2)].$$

which can be written as

$$\frac{c'(\tau)}{(1-p)(1+c'(\tau))} > \frac{\tau+c(\tau)}{1+c'(\tau)}[A(w_1)-A(w_2)][1-(1+c'(\tau))p].$$

6.3 Corollary 3.1

Since $u()$ is a concave function and $\tau < L$ we know that $u'(w_1) > u'(w_2)$. A sufficient condition for inequality (11) to be satisfied is

$$1-[1-(1+c'(\tau))p]\frac{\tau+c(\tau)}{1+c'(\tau)}A(w_1) > 1-(1-p)(1+c'(\tau))\frac{\tau+c(\tau)}{1+c'(\tau)}A(w_2)$$

This can also be written as

$$(1-p)(1+c'(\tau))A(w_2) > [1-(1+c'(\tau))p]A(w_1)$$

Using the expression 8 of the coefficient of relative risk aversion for a HARA function, we obtain

$$\frac{(1-p)(1+c'(\tau))}{1-(1+c'(\tau))p} > \frac{\eta+\frac{w_2}{\gamma}}{\eta+\frac{w_1}{\gamma}} \tag{12}$$

At an interior solution, the first order condition (5) yields

$$\frac{(1-p)(1+c'(\tau))}{1-(1+c'(\tau))p} = \Big(\frac{\eta+\frac{w_2}{\gamma}}{\eta+\frac{w_1}{\gamma}}\Big)^{\gamma}$$

So for any $\gamma > 1$, this implies that (12) is verified and optimal coverage is indeed decreasing in p.

6.4 Proposition 5

Assume that L is now a function of p such that a change in p is compensated by a change in L that maintain pL constant, i.e.

$$L'(p) = -\frac{L}{p} \tag{13}$$

Let τ^*, w_1^* and w_2^* be the optimal values of coverage, loss state wealth and no-loss state wealth. The first order condition now writes

$$[1-(1+c'(\tau^*))p]u'(w+\tau^*-p\tau^*-pc(\tau^*)-L(p)) = (1-p)(1+c'(\tau^*))u'(w-p\tau^*-pc(\tau^*)) \tag{14}$$

Proceeding as in the proof of Proposition 3, we find the necessary and sufficient condition for $\frac{d\tau^*}{dp} < 0$

$$\frac{c'(\tau^*)}{(1-p)[1-(1+c'(\tau^*))p]} > [\tau^* + c(\tau^*)][A(w_1^*) - A(w_2^*)] + A(w_1^*)L'(p) \tag{15}$$

Using equation 13

$$\begin{aligned}
[\tau^* + c(\tau^*)][A(w_1^*) &- A(w_2^*)] + A(w_1^*)L'(p) = [\tau^* + c(\tau^*)][A(w_1^*) - A(w_2^*)] - A(w_1^*)\frac{L}{p} \\
&= [\tau^* + c(\tau^*)]\Big(A(w_1^*) - A(w_2^*) - A(w_1^*)\frac{L}{[\tau^* + c(\tau^*)]p}\Big) \\
&= [\tau^* + c(\tau^*)]\frac{A(w_1^*)[\tau^* + c(\tau^*)]p - A(w_2^*)[\tau^* + c(\tau^*)]p - A(w_1^*)L}{[\tau^* + c(\tau^*)]p} \\
&= [\tau^* + c(\tau^*)]\frac{A(w_1^*)\{[\tau^* + c(\tau^*)]p - L\} - A(w_2^*)}{[\tau^* + c(\tau^*)]p}. \qquad (16)
\end{aligned}$$

At any interior solution, the left-hand side of 14 must be positive. Therefore, it must be the case that

$$p(1 + c'(\tau^*)) < 1.$$

By convexity of $c(\tau)$, we have

$$p(1 + c'(\tau)) < p(1 + c'(\tau^*)) < 1 \quad \forall \tau < \tau^*.$$

Hence

$$\int_0^{\tau^*} p(1 + c'(\tau))d\tau \;\; < \;\; \int_0^{\tau^*} d\tau.$$

Or equivalently, using $c(0) = 0$

$$p(\tau^* + c(\tau^*)) < \tau^* < L.$$

Therefore

$$A(w_1^*)\Big([\tau^* + c(\tau^*)]p - L\Big) < 0,$$

which, from 16 implies

$$[\tau^* + c(\tau^*)][A(w_1^*) - A(w_2^*)] + A(w_1^*)L'(p) < 0.$$

The necessary and sufficient condition 15 is therefore always satisfied.

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