Insurance demand under downside loss aversion *

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Abstract

Under Expected Utility it is often assumed that the utility function u is everywhere (continuously) differentiable.

In this paper we analyze the implications of a point of discontinuity of u' for optimal behavior. We also establish a link between discontinuity of u' and the presence of downside loss aversion.

1 Introduction

In a recent survey of the theory of insurance demand, Schlesinger [17] indicates at the beginning of the conclusion that "Mossin's theorem is often considered to be the cornerstone of modern insurance economics".

As is well known, Mossin's theorem [14] states that under Expected Utility (EU) an insured should buy less than full coverage when the premium is positively loaded. In a footnote attached to his conclusion, Schlesinger mentions that "this result depends on the differentiability of the von Neumann-Morgenstern utility function. "Kinks" in the utility function can lead to violations of Mossin's result". To the best of our knowledge this point was made for the first time by Segal and Spivak [18] who formally defined "first order risk aversion" (F.O.R.A.)[1]. First order risk aversion emerges when the utility function exhibits at least one point of non differentiability.

Segal and Spivak [18] apply the concept to insurance demand and they show that for *binary* risks full insurance may be optimal even in the presence of a loaded premium.

In the present paper we extend the results to a general distribution of the random loss. This extension provides new intuitions and interpretations about the implications of F.O.R.A. We also relate our analysis to the notion of downside loss aversion under EU[2] as discussed in Jarrow and Zhao [9].

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[1] See also Machina [12], especially section 3.7., and Schlesinger [16].

[2] In the literature downside loss aversion is usually presented in the context of cumulative prospect theory. See e.g. among many references Köbberling and Wakker [10], Bernard and Ghossoub [2] and the literature therein.

Much after Mossin's theorem another topic became popular in the insurance economics literature, namely the impact of an independent zero mean background risk on the demand for insurance (for early contributions, see Mayers and Smith [13] and Doherty and Schlesinger [6]). Again this literature was developed under the (implicit) assumption of second order risk aversion (S.O.R.A.), i.e. for utility functions that are everywhere differentiable.

Its main implication is that, under standard assumptions about the behavior of absolute risk aversion and prudence, the independent background risk should stimulate the demand for insurance.

We show in this paper that under F.O.R.A. just the opposite situation may prevail and we explain why. As a result it appears that there are important differences inside the EU model between F.O.R.A. and S.O.R.A. in terms of optimal behavior.

Our paper is organized as follows. In section 2 we present the model and its notation. We also show that F.O.R.A. can emerge inside EU as a consequence of downside loss aversion as defined in Jarrow and Zhao [9]. The optimality of full insurance coverage without background risk is discussed in section 3, while the impact of background risk is the subject of section 4. Our conclusions are drawn in section 5. Three appendices complete the work with (A) some additional results about the relationship between downside loss aversion and risk aversion below a target wealth, (B) a graphical analysis that illustrates the interpretation of the main results in a simple context, and (C) the impact of a point mass in the distribution of background risk on the optimal choice of insurance coverage.

2 The model and its notation

We consider a decision maker (D.M.) with initial wealth w_0 and a potential random loss L defined on $[0, \bar{l}]$ where $\bar{l}$ is the potential maximum loss with $\bar{l} \leq w_0$. We denote by $F_L(l)$ the cumulative distribution function of L and by μ its expected value.

The premium π charged for a proportional coverage α ($0 \leq \alpha \leq 1$) contains a loading factor λ ($\lambda \geq 0$) so that:

$$\pi = (1 + \lambda)\alpha\mu \tag{2.1}$$

Under F.O.R.A. the utility function is concave with a point of non differentiability. Such a situation may arise because of downside loss aversion inside the EU model as argued by Jarrow and Zhao [9]. In this model, the utility function $u(w)$ is defined as follows:

- above a target t, $u(w) = h(w)$ where $h(w)$ is increasing and concave;
- below the target t, $u(w) = h(w) + g(w)$ where $g(w)$ is such that:
    - $g(w) < 0$ for $w < t$ and $g(t) = 0$
    - $g'(w) > 0$, $g''(w) \leq 0$

In a sense the function $g(w)$ represents the pain (i.e. the loss of utility) due to the fact that the target has not been reached. Because both $h(w)$ and $g(w)$ are concave, the utility function is everywhere locally concave[3] except at the kink where the local index of absolute risk aversion is not defined. Notice that -to some extent surprisingly- the introduction of downside loss aversion has an ambiguous impact on the degree of local risk aversion below the target. This is shown in Appendix A.

For such models the choice of the target plays a central role. In a paper dealing with portfolio choice under cumulative prospect theory, Bernard and Ghossoub [2] convincingly argue that the target corresponds to "the amount the individual would have received at the end of the period had he invested all of his initial wealth in the risk-free asset".

In a sense in the portfolio choice problem the target corresponds to the choice of a zero risk situation. The equivalent target in an insurance problem corresponds to a situation where the D.M. is fully insured so that the target t is given by

$$t = w_0 - (1+\lambda)\mu. \tag{2.2}$$

3 The optimality of full coverage

In the absence of any background risk, the final wealth of a D.M. that insures a fraction[4] α of the potential loss L is given by:

$$W(L,\alpha) = w_0 - L + \alpha L - \pi(L) = w_0 - (1-\alpha)L - (1+\lambda)\alpha\mu$$

The D.M. chooses the optimal level of insurance coverage $\alpha^* \in [0,1]$ by maximizing the expected utility of his final wealth:

$$\begin{aligned} U(\alpha) = E[u(W)] &= \int_0^{\bar{l}} u\left[w(l,\alpha)\right] dF_L(l) \\ &= \int_0^{l^t} h\left[w(l,\alpha)\right] dF_L(l) + \int_{l^t}^{\bar{l}} \left\{h\left[w(l,\alpha)\right] + g\left[w(l,\alpha)\right]\right\} dF_L(l) \end{aligned} \tag{3.1}$$

where $l^t = (1+\lambda)\mu$ is the loss incurred at the target wealth t assumed in (4.5) (henceforth, the *target loss*), so that $l \leq l^t$ iff $w \geq t$.

Notice that, under F.O.R.A., the utility function u is not differentiable in $w = t$, hence for $L = l^t$. Nevertheless, assuming that L is continuous at l^t (together with Lebesgue's Dominated Convergence Theorem) guarantees that

$$U'(\alpha) = \int_0^{l^t} h'[w(l,\alpha)]w_\alpha dF_L(l) + \int_{l^t}^{\bar{l}} \left\{h'\left[w(l,\alpha)\right] + g'\left[w(l,\alpha)\right]\right\} w_\alpha dF_L(l) \tag{3.2}$$

where $w_\alpha = \frac{\partial w}{\partial \alpha} = l - (1+\lambda)\mu$.

Proposition 3.1. *Consider the insurance-purchasing decision of an EU-maximizing agent whose utility function exhibits F.O.R.A. at the target wealth t specified in* (4.5).

[3] Notice the difference with the non EU models of loss aversion mentioned in footnote 2 where $u(w)$ has convex portions. For historical interest, it is worth mentioning that a "curvilinear kinked utility function" of the type considered in the present paper was already proposed in 1975 by Greer and Skekel to model some experimental findings (see Figure 8, p. 843, in Greer and Skekel [8]).

[4] Since it is often the case that over insurance is not allowed, we maintain throughout the paper the assumption that the coinsurance rate α cannot exceed 1.

Define the upper and lower partial expectation of L with respect to the target loss $l^t = (1+\lambda)\mu$ as follows:

$$\mu_L^{-,t} = E\left(\max\left\{l^t - L, 0\right\}\right) = \int_0^{l^t} (l^t - l) dF_L(l)$$
$$\mu_L^{+,t} = E\left(\max\left\{L - l^t, 0\right\}\right) = \int_{l^t}^{\bar{l}} (l - l^t) dF_L(l)$$

where F_L is assumed continuous at l^t and $\mu_L^{+,t} - \mu_L^{-,t} = \mu - l^t = -\lambda\mu$. If:

$$\frac{h'(t) + g'(t)}{h'(t)} > \frac{\mu_L^{-,t}}{\mu_L^{+,t}} \tag{3.3}$$

then full coverage is optimal, i.e. $\alpha^* = 1$, even with a positive premium loading λ.

Proof. Under the F.O.R.A. hypothesis outlined in section 2, it is immediate to verify that the expected utility (3.1) is a concave function of the decision variable α for the target wealth t specified in (4.5). Consequently, the optimal level of insurance coverage α^* is found by studying the sign of:

$$U'(\alpha) = \int_0^{l^t} h'[w(l,\alpha)] w_\alpha dF_L(l) + \int_{l^t}^{\bar{l}} \left\{h'\left[w(l,\alpha)\right] + g'\left[w(l,\alpha)\right]\right\} w_\alpha dF_L(l).$$

Since $w(l,1) = w_0 - (1+\lambda)\mu = t$, evaluating $U'(\alpha)$ at $\alpha = 1$ gives:

$$\begin{aligned} U'(1) &= h'(t) \int_0^{l^t} w_\alpha dF_L(l) + [h'(t) + g'(t)] \int_{l^t}^{\bar{l}} w_\alpha dF_L(l) \\ &= -h'(t)\mu_L^{-,t} + [h'(t) + g'(t)]\mu_L^{+,t} \end{aligned} \tag{3.4}$$

which is strictly positive under assumption (3.3) since $\mu_L^{-,t}, \mu_L^{+,t} \geq 0$. It follows that $\alpha^* = 1$ is a constrained optimum for this case, when over insurance is prohibited. □

Remark 3.2. Under the general definition of loss aversion as the behavioral phenomenon that losses matter more than gains, the ratio $\frac{h'(t)+g'(t)}{h'(t)}$ which determines the occurrence of condition (3.3) may be interpreted as an index of loss aversion in the style of Kobberling and Wakker [10]. It is the magnitude of this index that can induce a rational, EU-maximizing agent to purchase full insurance coverage at a loaded price, in violation of Mossin's theorem but in accordance with common observation (see, e.g., Borch [4]).

Our result shows that the "amount" of loss aversion required to trigger the realization of (3.3) is determined by the partial moment ratio $\frac{\mu_L^{-,t}}{\mu_L^{+,t}}$, which compares the D.M.'s upside potential (the expectation $\mu_L^{-,t}$ of losses below the threshold l^t) to his/her downside risk exposure (the expectation $\mu_L^{+,t}$ of losses above l^t). This moment ratio is the analogue of a portfolio performance index that is well known in the quantitative finance literature as the Omega measure (see, e.g., Bertrand and Prigent [3] and the references therein).

4 Impact of background risk

So far we have assumed that the decision maker faces a single source of endogenous risk L, that he can avoid by choosing $\alpha^* = 1$. Economic decisions, however, often take place in the presence of one or more additional sources of risk, called "background risks", that can be neither controlled nor insured. Typical examples are risks associated to human capital, for which one may not find ready markets for risk sharing.

In a recent paper, Dionne and Li [5] have shown that EU exhibits F.O.R.A. in the presence of a background risk that is *positive expectation dependent* on the endogenous risk. Their model relies on the notion of conditional F.O.R.A. introduced by Loomes and Segal [11] and provides new insights into some economic and finance real world situations, such as the equity premium puzzle and the welfare cost of business cycles.

Our contribution differs from the precedent papers in that we focus attention on the impact of an *independent* background risk and we show that it can actually reduce the optimal insurance coverage of an EU agent under F.O.R.A.

In the following, we represent background risk by a random variable ε with cumulative distribution function M_ε valued in a finite interval $[\underline{\varepsilon}, \overline{\varepsilon}]$ and independent of L. In the presence of background risk, the final wealth of a D.M. that insures a fraction α of the potential loss L is given by:

$$W_{BR} = W_{BR}(L, \alpha, \varepsilon) = w_0 - (1-\alpha)L - (1+\lambda)\alpha\mu + \varepsilon$$

The D.M.'s objective is to choose $\alpha^*_{BR} \in [0,1]$ so as to maximize his/her expected utility:

$$V(\alpha) = E[u(W_{BR})] = \int_0^{\overline{l}} \int_{\underline{\varepsilon}}^{\overline{\varepsilon}} u[w_{BR}(l, \alpha, \varepsilon)] dF_L(l) dM_\varepsilon(\varepsilon) \tag{4.1}$$

where the expectation operator is evaluated with respect to L and ε because the decision on the optimal insurance coverage must be made before the realization of both sources of risk.

Notice that, under F.O.R.A., the utility function u is not differentiable in $w = t$, hence for $\varepsilon = \varepsilon^t$ where $\varepsilon^t \triangleq (1-\alpha)[l - (1+\lambda)\mu]$. Nevertheless, assuming that either (i) M_ε is absolutely continuous, or (ii) L has a discrete distribution $\{(l_j, f_j), 1 \le j \le k : \sum_{j=1}^k f_j = 1\}$ and M_ε has no point masses at $\{l_j - (1+\lambda)\mu\}$ for $1 \le j \le k$, guarantee, in view of the Dominated Convergence Theorem, that

$$V'(\alpha) = \int_0^{\overline{l}} \left\{ \int_{\underline{\varepsilon}}^{\varepsilon^t} [h'(w_{BR}) + g'(w_{BR})]\, w_\alpha dM_\varepsilon(\varepsilon) + \int_{\varepsilon^t}^{\overline{\varepsilon}} h'(w_{BR}) w_\alpha dM_\varepsilon(\varepsilon) \right\} dF_L(l) \tag{4.2}$$

for any $\alpha < 1$, where $w_\alpha = \frac{\partial w_{BR}}{\partial \alpha} = l - (1+\lambda)\mu$.

Proposition 4.1. *Consider an EU-maximizing D.M. whose utility function exhibits F.O.R.A. at the target wealth t specified in* (4.5)*. The D.M. wants to choose the optimal level of insurance coverage $\alpha^*_{BR} \in [0,1]$ for the endogenous risk L in the presence of an independent background risk ε whose cumulative distribution function M_ε is valued in a finite interval $[\underline{\varepsilon}, \overline{\varepsilon}]$, with $\underline{\varepsilon} \le 0$ and $\overline{\varepsilon} \ge 0$. If any of the following conditions hold:*

(i) *M_ε is absolutely continuous,*

(ii) *L has a discrete distribution $\{(l_j, f_j), 1 \le j \le k : \sum_{j=1}^k f_j = 1\}$ and M_ε has no point masses at $\{0\}$ and at $\{l_j - (1+\lambda)\mu\}$ for $1 \le j \le k$,*

then the D.M. will choose partial coverage, i.e. $\alpha^*_{BR} < 1$.

Proof. Combining (4.1) with the definition of F.O.R.A. given in section 2, the expected utility of the insured agent can be written as:

$$\begin{aligned} V(\alpha) &= \int_0^{\bar{l}} \left\{ \int_{\underline{\varepsilon}}^{\varepsilon^t} \{h[w_{BR}(l,\alpha,\varepsilon)] + g[w_{BR}(l,\alpha,\varepsilon)]\} \, dM_\varepsilon(\varepsilon) \right. \\ &+ \left. \int_{\varepsilon^t}^{\bar{\varepsilon}} h[w_{BR}(l,\alpha,\varepsilon)] dM_\varepsilon(\varepsilon) \right\} dF_L(l) \end{aligned} \tag{4.3}$$

where $\varepsilon^t = (1-\alpha)[l-(1+\lambda)\mu]$ is the amount background risk at the target wealth t specified in (4.5) for a realization l of the insurable loss L. Since (4.3) is a concave function of the decision variable α, we can determine the optimal coinsurance rate by evaluating the first derivative of $V(\alpha)$:

$$\begin{aligned} V'(\alpha) &= \int_0^{\bar{l}} \left\{ \int_{\underline{\varepsilon}}^{\varepsilon^t} [h'(w_{BR}) + g'(w_{BR})] \, w_\alpha dM_\varepsilon(\varepsilon) - m_\varepsilon(\varepsilon^t)[h(t) + g(t)](l - (1+\lambda)\mu) \right. \\ &+ \left. \int_{\varepsilon^t}^{\bar{\varepsilon}} h'(w_{BR}) w_\alpha dM_\varepsilon(\varepsilon) + m_\varepsilon(\varepsilon^t) h(t)(l-(1+\lambda)\mu) \right\} dF_L(l) \\ &= \int_0^{\bar{l}} \left\{ \int_{\underline{\varepsilon}}^{\varepsilon^t} [h'(w_{BR}) + g'(w_{BR})] \, w_\alpha dM_\varepsilon(\varepsilon) + \int_{\varepsilon^t}^{\bar{\varepsilon}} h'(w_{BR}) w_\alpha dM_\varepsilon(\varepsilon) \right\} dF_L(l) \end{aligned}$$

where m_ε denotes the probability density function of ε and the last equality is due to the assumption that $g(t) = 0$ under F.O.R.A. (cf. section 2). Evaluating this expression at $\alpha = 1$ gives:

$$\begin{aligned} V'(1) &= \int_0^{\bar{l}} \left\{ \int_{\underline{\varepsilon}}^{0} [h'(t+\varepsilon) + g'(t+\varepsilon)] \, (l-(1+\lambda)\mu) dM_\varepsilon(\varepsilon) \right. \\ &+ \left. \int_0^{\bar{\varepsilon}} h'(t+\varepsilon)(l-(1+\lambda)\mu) dM_\varepsilon(\varepsilon) \right\} dF_L(l) \\ &= \left[\int_0^{\bar{l}} (l-(1+\lambda)\mu) dF_L(l) \right] \left[\int_{\underline{\varepsilon}}^{0} [h'(t+\varepsilon) + g'(t+\varepsilon)] dM_\varepsilon(\varepsilon) + \int_0^{\bar{\varepsilon}} h'(t+\varepsilon) dM_\varepsilon(\varepsilon) \right] \\ &= -\lambda\mu \left[\int_{\underline{\varepsilon}}^{0} [h'(t+\varepsilon) + g'(t+\varepsilon)] dM_\varepsilon(\varepsilon) + \int_0^{\bar{\varepsilon}} h'(t+\varepsilon) dM_\varepsilon(\varepsilon) \right] \end{aligned}$$

which is strictly negative because the premium loading λ is assumed positive, h and g are increasing, and $\mu = E(L) > 0$ (cf. section 2). It follows that $\alpha^*_{BR} < 1$, i.e. partial insurance is optimal so that the introduction of an independent background risk stimulates risk taking by the risk averse D.M. □

Since this result is in sharp contrast with the dominant view under S.O.R.A. that the background risk and the foreground one are substitutes, let's try now to give the intuition behind the result of Proposition 4.1.

To do so let's consider a situation where the background risk and the foreground one are binary random variables. Let initial wealth $w_0 = 1000$, while the distribution of the potential loss L is given by:

$$
\begin{array}{ccc}
 & & 0 \\
 & \nearrow^{\frac{1}{2}} & \\
L & & \\
 & \searrow_{\frac{1}{2}} & \\
 & & 400.
\end{array}
$$

If the loading factor is $\lambda = 0.5$, we have:

$$\pi = (1 + 0.5)\, 200\alpha = 300\alpha \tag{4.4}$$

$$t = 1000 - (1 + 0.5)\, 200 = 700 \tag{4.5}$$

so that under full insurance the target $t = 700$ is obtained with certainty.

Suppose that, in the absence of background risk, the h and g functions are such that $\alpha^* = 1$. By this choice the D.M. reveals that he accepts to loose money on average in order never to fall below the target. Indeed, if he had chosen $\alpha = 0.9$ he would have faced the lottery:

$$
\begin{array}{ccc}
 & & 600 + 360 - 270 = 690 \\
 & \nearrow^{\frac{1}{2}} & \\
W & & \\
 & \searrow_{\frac{1}{2}} & \\
 & & 1000 - 270 = 730.
\end{array}
$$

In fact under less than full insurance, the probability of falling below the target is 1/2. Now consider the introduction of a binary zero mean background risk ε given by:

$$
\begin{array}{ccc}
 & & -20 \\
 & \nearrow^{\frac{1}{2}} & \\
\varepsilon & & \\
 & \searrow_{\frac{1}{2}} & \\
 & & +20.
\end{array}
$$

The situation of full insurance -that was the most attractive in the absence of background risk- is now much deteriorated since with a probability of 1/2 final wealth will be below the target. It is easy to understand that less than full insurance can protect the individual against such a situation. Indeed the distribution of final wealth either under $\alpha = 1$ or under $\alpha = 0.9$ is given by:

$$
\begin{array}{ccc}
 & & 680 \\
 & \nearrow^{\frac{1}{2}} & \\
W & & \\
 & \searrow_{\frac{1}{2}} & \\
 & & 720
\end{array}
$$

for $\alpha = 1$, and

$$
\begin{array}{ccc}
 & & 670 \\
 & \nearrow^{\frac{1}{4}} & \\
W & \rightarrow_{\frac{1}{2}} & 710 \\
 & \searrow_{\frac{1}{4}} & \\
 & & 750
\end{array}
$$

for $\alpha = 0.9$. The solution $\alpha = 0.9$ may be much more attractive than that of $\alpha = 1$ not only because it generates a higher expected wealth but also because it induces a sharp reduction of the probability of falling below the target (1/4 instead of 1/2). While it creates additional risk of the second order, the background risk combined with less than full insurance also protects against the worst scenario, i.e. a wealth below the target. In Appendix B more computational details are provided.

Notice that the absence of a point mass at $\{0\}$ in the distribution of background risk is a key assumption in the derivation of the previous result. Different distributional assumptions can actually induce an EU agent to maintain full insurance coverage under F.O.R.A., as shown in Appendix C.

5 Conclusion

The concept of F.O.R.A. introduced by Segal and Spivak [18] has so far received much less attention than that of S.O.R.A. which is dominant in the economics of risk literature.

We have shown in this paper that nevertheless F.O.R.A. might be an attractive concept because it is directly related under EU to the notion of downside loss aversion as defined by Jarrow and Zhao [9]. By considering a general random loss (instead of a binary one as in Segal and Spivak [18]) we have shown that full insurance may be optimal even when the premium is loaded. This happens when an index of the intensity of loss aversion exceeds the statistical measure of the importance of downside risk[5].

When background risks are considered another important difference between F.O.R.A. and S.O.R.A. also emerges. Under S.O.R.A. it is widely believed that background and foreground risks are substitutes because this result emerges under standard assumptions made about the shape of the utility function (e.g. decreasing prudence). On the contrary, under F.O.R.A. it may happen that the background and foreground risks are complements, especially when the D.M. is fully insured before the introduction of the background risk.

A Appendix 1

In the following lemma we characterize the relationship between downside loss aversion inside the EU model and risk aversion below the target wealth.

Lemma A.1. *Let h be an increasing, concave and (at least) twice differentiable function of w and define the utility function under F.O.R.A. as:*

$$u(w) = \begin{cases} h(w) + g(w); & \text{if } w < t \\ h(w); & \text{if } w \geq t \end{cases} \tag{A.1}$$

where g is such that: $g(t) = 0$, $g(w) < 0$, $g'(w) > 0$ and $g''(w) \leq 0$ for $w < t$. On the interval $(-\infty, t)$, u is more risk averse than h if and only if g is more risk averse than h.

[5]This comparison is reminiscent of the one often made under S.O.R.A. between the index of absolute risk aversion and the measure of the quantity of risk faced by the D.M.

Proof. By definition of the Arrow-Pratt coefficient of absolute risk aversion (ARA), u is more risk averse than h on the interval $(-\infty, t)$ iff

$$ARA_u(w) = -\frac{u''(w)}{u'(w)} \geq -\frac{h''(w)}{h'(w)} = ARA_h(w)$$

for $w < t$. Using (A.1), we can rewrite this condition as:

$$-\frac{h''(w) + g''(w)}{h'(w) + g'(w)} \geq -\frac{h''(w)}{h'(w)}.$$

Since $h', g' > 0$, this is equivalent to:

$$\begin{aligned} -h'(w)\left[h''(w) + g''(w)\right] &\geq -h''(w)\left[h'(w) + g'(w)\right] \\ -h'(w)\, g''(w) &\geq -h''(w)\, g'(w) \\ -\frac{g''(w)}{g'(w)} &\geq -\frac{h''(w)}{h'(w)} \end{aligned}$$

that is, $ARA_g(w) \geq ARA_h(w)$. □

We conclude that the introduction of downside loss aversion does not necessarily increase risk aversion below the target.

B Appendix 2

In view of explaining the intuition behind Proposition 4.1 we here provide some pictures that illustrate the numerical example on p. 6 and 7 for a quadratic function $h(w) = w - \beta w^2$ that is altered by a linear plus exponential (linex) component $g(w) = (w - t) - e^{-\gamma(w-t)} + 1$ when final wealth is below the target t. Our utility function is thus given by:

$$u(w) = \begin{cases} w - \beta w^2 + (t - w) - e^{\gamma(t-w)} + 1; & \text{if } w < t \\ w - \beta w^2; & \text{if } w \geq t \end{cases} \tag{B.1}$$

where $0 < \beta < \frac{1}{2w}$ and $\gamma > 0$.

Based on $w_0 = 1000$, on the binary loss L specified on p. 7 and on the values of π and t given in (4.4) and (4.5) respectively, we here choose $\beta = 0.000488$ and $\gamma = 0.05$ in view of fulfilling condition (3.3). We thus obtain the utility function represented in Figure 1.

In the absence of any background risk, condition (3.3) implies that Expected Utility is maximized at $\alpha = 1$ as shown in Figure 2. In the same Figure we illustrate the impact of a binary background risk ε, independent of L, that takes values -20 and $+20$ with equal probabilities (p. 7). The solution $\alpha = 0.9$ is now more attractive than that of $\alpha = 1$, as explained on p. 8.

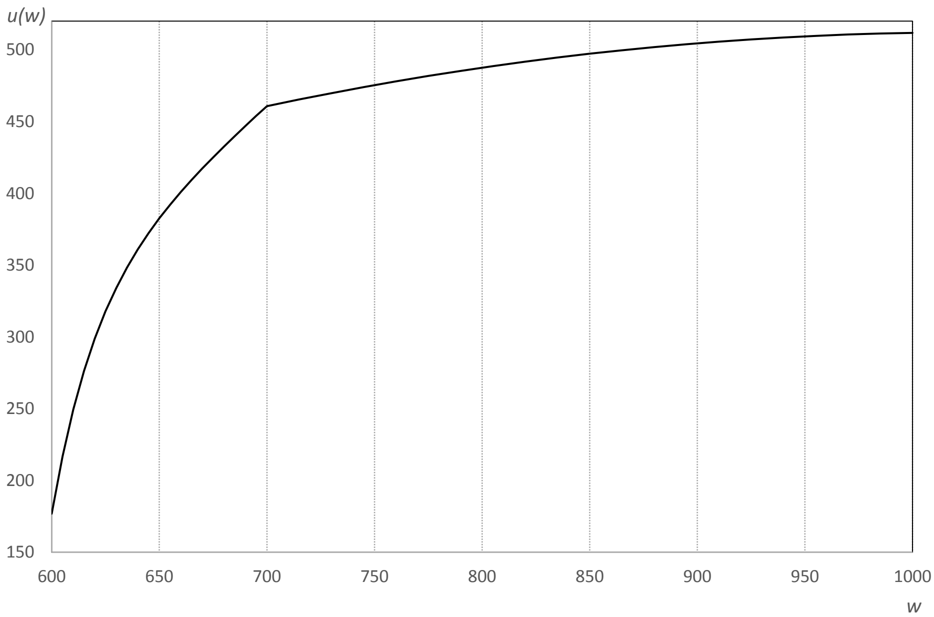

Figure 1: The utility function (B.1) with target wealth $t = 700$. This utility function is quadratic (with $\beta = 0.000488$) for $w > t$, and quadratic plus linex (with $\gamma = 0.05$) for $w \leq t$.

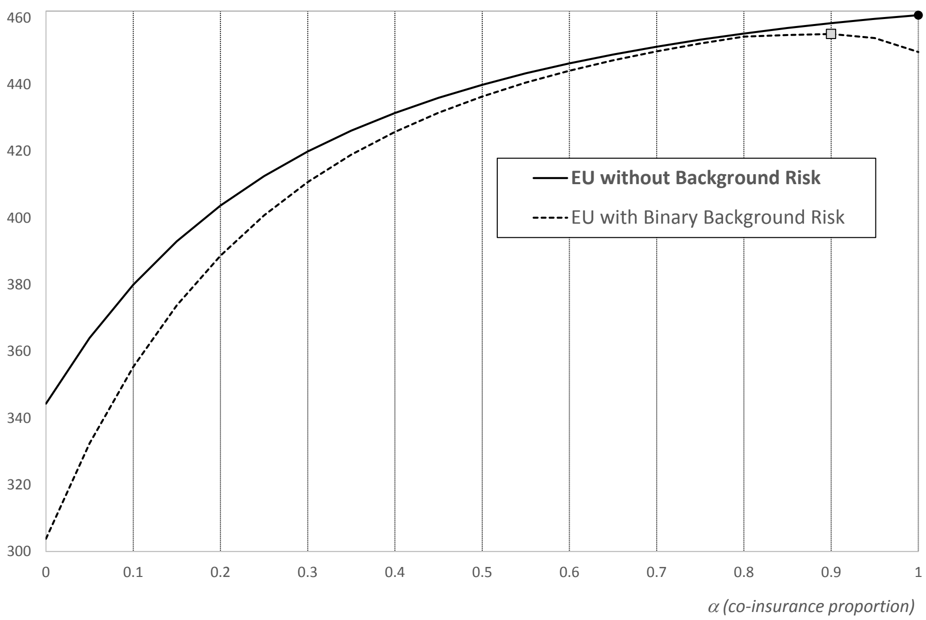

Figure 2: Expected Utility as a function of the coverage rate α.

C Appendix 3

We here extend the framework of Proposition 4.1 by introducing a point mass at $\{0\}$ in the distribution of background risk. The impact of this stochastic change on the optimal insurance coverage is characterized in the following proposition.

Proposition C.1. *Consider an EU-maximizing D.M. whose utility function exhibits F.O.R.A. at the target wealth t specified in* (4.5)*. The D.M. wants to choose the optimal level of insurance coverage $\alpha^*_{BR} \in [0,1]$ for the endogenous risk L in the presence of an independent background risk ε whose cumulative distribution function is denoted by M_ε. Suppose that M_ε fulfills the assumptions of Proposition 4.1 except that at the value $\varepsilon = \{0\}$ a probability mass $P_\varepsilon(0) = Prob(\varepsilon = 0) > 0$ is now allowed.*

1. *If the support of M_ε is the interval $[0, \overline{\varepsilon}]$ with $\overline{\varepsilon} > 0$, then $\alpha^*_{BR} = 1$ if:*

$$\frac{g'(t)}{h'(t)} \cdot P_\varepsilon(0) > \frac{\lambda\mu}{\mu_L^{+,t}}. \tag{C.1}$$

2. *If the support of M_ε is the interval $[\underline{\varepsilon}, 0]$ with $\underline{\varepsilon} < 0$, then $\alpha^*_{BR} = 1$ if:*

$$\frac{g'(t) \cdot P_\varepsilon(0)}{\int_{[\underline{\varepsilon},0]}[h'(t+\varepsilon) + g'(t+\varepsilon)]dM_\varepsilon(\varepsilon) - g'(t)P_\varepsilon(0)} > \frac{\lambda\mu}{\mu_L^{+,t}}. \tag{C.2}$$

3. *If the support of M_ε is the interval $[\underline{\varepsilon}, \overline{\varepsilon}]$ with $\underline{\varepsilon} < 0$ and $\overline{\varepsilon} > 0$, then $\alpha^*_{BR} = 1$ if:*

$$\frac{g'(t)P_\varepsilon(0)}{h'(t)P(\varepsilon > 0) - g'(t)P_\varepsilon(0) + \int_{[\underline{\varepsilon},0]}[h'(t+\varepsilon) + g'(t+\varepsilon)]dM_\varepsilon(\varepsilon)} > \frac{\lambda\mu}{\mu_L^{+,t}}. \tag{C.3}$$

Proof. We give here only the proof of item 1. The proofs of items 2 and 3 are available from the authors upon request.

1. Suppose that ε is valued in $[0, \overline{\varepsilon}]$, with $\overline{\varepsilon} > 0$, and a probability mass at 0 is allowed, i.e. $P_\varepsilon(0) = Prob(\varepsilon = 0) > 0$. Then

$$\begin{aligned} V(\alpha) &= P_\varepsilon(0) \overbrace{\int_0^{\overline{l}} u[w(l,\alpha)]dF_L(l)}^{U(\alpha)} + \\ &+ \int_0^{(1+\lambda)\mu} \left\{ \int_{(0,\overline{\varepsilon}]} h[w_{BR}(l,\alpha,\varepsilon)]dM_\varepsilon(\varepsilon) \right\} dF_L(l) \\ &+ \int_{(1+\lambda)\mu}^{\overline{l}} \left\{ \int_{(0,\varepsilon^t]} (h+g)[w_{BR}(l,\alpha,\varepsilon)]dM_\varepsilon(\varepsilon) + \int_{\varepsilon^t}^{\overline{\varepsilon}} h[w_{BR}(l,\alpha,\varepsilon)]dM_\varepsilon(\varepsilon) \right\} dF_L(l) \end{aligned}$$

where $U(\alpha)$ is the expected utility without background risk defined in (3.1).

Hence

$$
\begin{aligned}
V'(\alpha) &= P_\varepsilon(0)U'(\alpha) \\
&+ \int_0^{(1+\lambda)\mu} \left\{ \int_{(0,\bar{\varepsilon}]} h'(w_{BR}) w_\alpha dM_\varepsilon(\varepsilon) \right\} dF_L(l) \\
&+ \int_{(1+\lambda)\mu}^{\bar{l}} \left\{ \int_{(0,\varepsilon^t]} (h'+g')(w_{BR}) w_\alpha dM_\varepsilon(\varepsilon) + [h(t)+g(t)](l-(1+\lambda)\mu) m_\varepsilon(\varepsilon^t) \right. \\
&\quad \left. + \int_{\varepsilon^t}^{\bar{\varepsilon}} h'(w_{BR}) w_\alpha dM_\varepsilon(\varepsilon) - h(t)(l-(1+\lambda)\mu) m_\varepsilon(\varepsilon^t) \right\} dF_L(l) \\
&= P_\varepsilon(0)U'(\alpha) + \int_0^{(1+\lambda)\mu} \left\{ \int_{(0,\bar{\varepsilon}]} h'(w_{BR}) w_\alpha dM_\varepsilon(\varepsilon) \right\} dF_L(l) \\
&+ \int_{(1+\lambda)\mu}^{\bar{l}} \left\{ \int_{(0,\varepsilon^t]} (h'+g')(w_{BR}) w_\alpha dM_\varepsilon(\varepsilon) + \int_{\varepsilon^t}^{\bar{\varepsilon}} h'(w_{BR}) w_\alpha dM_\varepsilon(\varepsilon) \right\} dF_L(l).
\end{aligned}
$$

Evaluating this expression at $\alpha = 1$ and using (3.4) and concavity of h give:

$$
\begin{aligned}
V'(1) &= P_\varepsilon(0)U'(1) + \int_0^{(1+\lambda)\mu} \left\{ \int_{(0,\bar{\varepsilon}]} h'(t+\varepsilon)(l-(1+\lambda)\mu) dM_\varepsilon(\varepsilon) \right\} dF_L(l) \\
&+ \int_{(1+\lambda)\mu}^{\bar{l}} \left\{ \int_{(0,\bar{\varepsilon}]} h'(t+\varepsilon)(l-(1+\lambda)\mu) dM_\varepsilon(\varepsilon) \right\} dF_L(l) \\
&= P_\varepsilon(0)U'(1) - \lambda\mu \left[\int_{(0,\bar{\varepsilon}]} h'(t+\varepsilon) dM_\varepsilon(\varepsilon) \right] \\
&\geq P_\varepsilon(0)U'(1) - \lambda\mu h'(t) \cdot P(\varepsilon > 0) \\
&= P_\varepsilon(0) \cdot \left\{ -h'(t)\mu_L^{-,t} + [h'(t)+g'(t)]\mu_L^{+,t} \right\} - \lambda\mu h'(t) \cdot P(\varepsilon > 0) \\
&= h'(t) \left\{ P_\varepsilon(0) \left[-\mu_L^{-,t} + \mu_L^{+,t} \right] - \lambda\mu P(\varepsilon > 0) \right\} + g'(t)\mu_L^{+,t} P_\varepsilon(0) \\
&= h'(t) \left\{ -\lambda\mu P_\varepsilon(0) - \lambda\mu P(\varepsilon > 0) \right\} + g'(t)\mu_L^{+,t} P_\varepsilon(0) \\
&= -\lambda\mu h'(t) + g'(t)\mu_L^{+,t} P_\varepsilon(0)
\end{aligned}
$$

which is strictly positive under condition (C.1).

□

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